# Oscilador Armónico III

Recapitulación

$$\boxed{\begin{aligned} a^{\dagger}|\phi_n\rangle &= \sqrt{n+1}\,|\phi_{n+1}\rangle \\ a|\phi_n\rangle &= \sqrt{n}\,|\phi_{n-1}\rangle \end{aligned}}$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

- Definimos $a$ (descenso) y $a^{+}$ (ascenso); $[a, a^{+}] = 1$

$[N, a] = -a$

$[N, a^{+}] = a^{+}$

- $N = a^{+}a \neq aa^{+}$ (o, o)

$N$ es hermitiano; $H = \hbar\omega(N + \frac{1}{2})$

$N|\phi_n\rangle = n|\phi_n\rangle$ $\quad n \geq 0$ entero

$H = \hbar\omega(n + \frac{1}{2})$

$a|\phi_0\rangle = 0$

$a|\phi_n\rangle$ e-vector de $N$ con e-valor $n-1$

$a^{+}|\phi_n\rangle$ e-vector de $N$ con e-valor $n+1$

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{X} + i\frac{1}{\sqrt{m\omega\hbar}}\hat{P}\right)$$

$$a^{+} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{X} - i\frac{1}{\sqrt{m\omega\hbar}}\hat{P}\right)$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^n|\phi_0\rangle$$

En la representación $\{|x\rangle\}$

$$\phi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Los demás estados se obtienen de este:

$$\varphi_n(x) = \langle x|\varphi_n\rangle = \frac{1}{\sqrt{n!}}\langle x|(a^{\dagger})^n|\varphi_0\rangle$$

$$= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{2^n}}\left[\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{\mathrm{d}}{\mathrm{d}x}\right]^n \varphi_0(x)$$

that is:

$$\varphi_n(x) = \left[\frac{1}{2^n n!}\left(\frac{\hbar}{m\omega}\right)^n\right]^{1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\frac{m\omega}{\hbar}x - \frac{\mathrm{d}}{\mathrm{d}x}\right]^n \mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \qquad \text{(C-27)}$$

It is easy to see from this expression that $\varphi_n(x)$ is the product of $\mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$ and a polynomial of degree $n$ and parity $(-1)^n$, called a *Hermite polynomial* (*cf.* Complements $B_V$ and $C_V$).

A simple calculation gives the first several functions $\varphi_n(x)$:

$$\varphi_1(x) = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} x\,\mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$$\varphi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left[2\frac{m\omega}{\hbar}x^2 - 1\right]\mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \qquad \text{(C-28)}$$

Calculamos $\phi_0(x)$ y $\phi_1(x)$ la clase pasada así que no es necesario hacer las cuentas.

Podríamos hacerlo también en la representación $\{|p\rangle\}$ usando $\langle p|\hat{X}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\psi(p)$ $\quad \langle p|\hat{P}|\psi\rangle = p\,\psi(p)$

$$a|\phi_0\rangle = 0 \Rightarrow \langle p|a|\phi_0\rangle = \langle p|\left(\sqrt{\frac{m\omega}{\hbar}}\hat{X} + i\frac{1}{\sqrt{m\omega\hbar}}\hat{P}\right)|\phi_0\rangle = 0$$ (tarea)

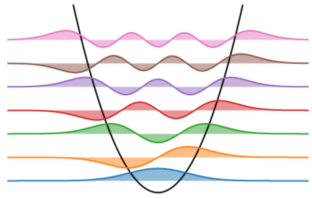
[poner compu] mostrar sección de Eigenfunciones

Observaciones (Cohen-Tannoudji V. C-2)

al aumentar n

- más ancha ⇒ más energía potencial promedio
- más ceros ⇒ más energía cinética porque hay más curvatura y $\frac{d^2}{dx^2}\phi_n$ crece

$$\frac{1}{2m}\langle p^2\rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\phi_n^*(x)\frac{d^2}{dx^2}\phi_n(x)\,dx$$



- Más densidad en los extremos igual que el caso clásico.

#### D-1. Mean values and root mean square deviations of $X$ and $P$ in a state $|\varphi_n\rangle$

Neither $X$ nor $P$ commutes with $H$, and the eigenstates $|\varphi_n\rangle$ of $H$ are not eigenstates of $X$ or $P$. Consequently, if the harmonic oscillator is in a stationary state $|\varphi_n\rangle$, a measurement of the observable $X$ or the observable $P$ can, *a priori*, yield any result (since the spectra of $X$ and $P$ include all real numbers). We shall now calculate the mean values of $X$ and $P$ in such a stationary state and then their root mean square deviations $\Delta X$ and $\Delta P$, which will enable us to verify the uncertainty relation.

As we indicated in § C-1-c, we shall perform these calculations with the help of the operators $a$ and $a^\dagger$. As far as the mean values of $X$ and $P$ are concerned, the result follows directly from formulas (C-22), which show that neither $X$ nor $P$ has diagonal matrix elements:

$$\begin{aligned}
&\langle\varphi_n|X|\varphi_n\rangle = 0\\
&\langle\varphi_n|P|\varphi_n\rangle = 0
\end{aligned} \tag{D-1}$$

To obtain the root mean square deviations $\Delta X$ and $\Delta P$, we must calculate the mean values of $X^2$ and $P^2$:

$$\begin{aligned}
(\Delta X)^2 &= \langle\varphi_n|X^2|\varphi_n\rangle - (\langle\varphi_n|X|\varphi_n\rangle)^2 = \langle\varphi_n|X^2|\varphi_n\rangle\\
(\Delta P)^2 &= \langle\varphi_n|P^2|\varphi_n\rangle - (\langle\varphi_n|P|\varphi_n\rangle)^2 = \langle\varphi_n|P^2|\varphi_n\rangle
\end{aligned} \tag{D-2}$$

Now, according to (B-1) and (B-7):

$$\begin{aligned}
X^2 &= \frac{\hbar}{2m\omega}(a^\dagger + a)(a^\dagger + a)\\
&= \frac{\hbar}{2m\omega}(a^{\dagger 2} + aa^\dagger + a^\dagger a + a^2)\\
P^2 &= -\frac{m\hbar\omega}{2}(a^\dagger - a)(a^\dagger - a)\\
&= -\frac{m\hbar\omega}{2}(a^{\dagger 2} - aa^\dagger - a^\dagger a + a^2)
\end{aligned} \tag{D-3}$$

The terms in $a^2$ and $a^{\dagger 2}$ do not contribute to the diagonal matrix elements, since $a^2|\varphi_n\rangle$ is proportional to $|\varphi_{n-2}\rangle$, and $a^{\dagger 2}|\varphi_n\rangle$ to $|\varphi_{n+2}\rangle$; both are orthogonal to $|\varphi_n\rangle$. On the other hand:

$$\begin{aligned}\langle\varphi_n|(a^\dagger a + aa^\dagger)|\varphi_n\rangle &= \langle\varphi_n|(2a^\dagger a + 1)|\varphi_n\rangle \\ &= 2n+1\end{aligned} \tag{D-4}$$

Consequently:

$$(\Delta X)^2 = \langle\varphi_n|X^2|\varphi_n\rangle = \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega} \tag{D-5a}$$

$$(\Delta P)^2 = \langle\varphi_n|P^2|\varphi_n\rangle = \left(n+\frac{1}{2}\right)m\hbar\omega \tag{D-5b}$$

Propiedades de $|\phi_0\rangle$

- $E \neq 0$ a diferencia del caso clásico.
- Tiene extensión espacial
- Balance $T$ y $U$ + principio de incertidumbre

# Evolución temporal

### D-3. Time evolution of the mean values

Consider a harmonic oscillator whose state at $t = 0$ is:

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n(0)|\varphi_n\rangle \tag{D-22}$$

($|\psi(0)\rangle$ is assumed to be normalized). Its state $|\psi(t)\rangle$ at $t$ can be obtained by using rule (D-54) of Chapter III:

$$\begin{aligned}|\psi(t)\rangle &= \sum_{n=0}^{\infty} c_n(0)\,\mathrm{e}^{-i\;E_n t/\hbar}|\varphi_n\rangle \\ &= \sum_{n=0}^{\infty} c_n(0)\,\mathrm{e}^{-i\left(n+\frac{1}{2}\right)\omega t}|\varphi_n\rangle\end{aligned} \tag{D-23}$$

The mean value of any physical quantity $A$ is therefore given as a function of time by:

$$\langle\psi(t)|A|\psi(t)\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^*(0)\;c_n(0)\;A_{mn}\,\mathrm{e}^{i(m-n)\omega t} \tag{D-24}$$

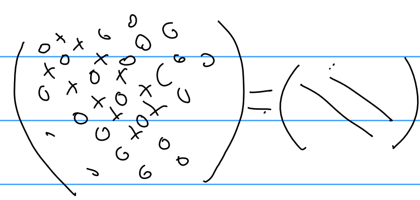
with:

$$A_{mn} = \langle\varphi_m|A|\varphi_n\rangle \tag{D-25}$$

frecuencia $\omega$ y armóicos

$A = X$

$$\langle\varphi_m|X|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right]$$



No quedan los armónicos

$P$ también tiene una forma similar

$\therefore$ En $\langle P\rangle(t)$ y $\langle X\rangle(t)$ sólo aparecen términos que oscilan como $\omega t$.

# Ejemplos de evolución temporal

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} C_n(0)\, e^{-i\omega(n+\frac{1}{2})t}\,|\varphi_n\rangle$$

→ Si $C_{n'}(0) = 1$ y los demás $C_n(0) = 0$

$$|\Psi(t)\rangle = e^{-i\omega(n'+\frac{1}{2})t}\,|\varphi_{n'}\rangle$$

$$\Psi(x,t) = e^{-i\omega(n'+\frac{1}{2})t}\,\varphi_{n'}(x)$$

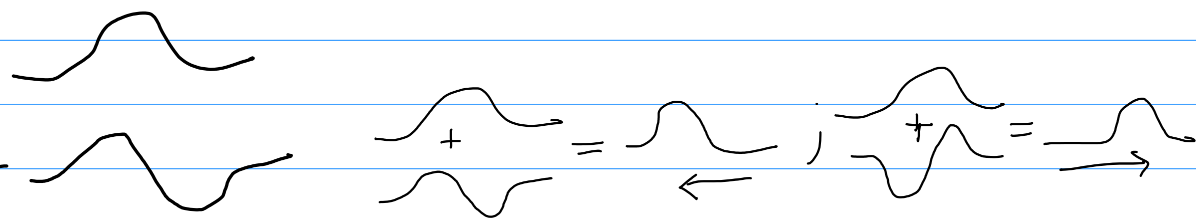
∴ Si el estado inicial es un e-estado de $\hat{H}$, no cambia la forma de $\Psi(x,t)$ con el tiempo.

→ $C_0(0) = \frac{1}{\sqrt{2}}$, $C_1(0) = \frac{1}{\sqrt{2}}$, los demás son cero.

$$|\Psi(t)\rangle = \frac{e^{-i\frac{\omega}{2}t}|\varphi_0\rangle + e^{-i\frac{3\omega}{2}t}|\varphi_1\rangle}{\sqrt{2}}$$

$\varphi_0 =$

$-\varphi_1 =$



[poner compu]

mostrar sección de evolucion temporal.